Homework 4: Solutions to exercises not appearing in Pressley, also 2.3.2, 2.3.4, and 2.3.5

Math 120A

- (2.3.2) We already know that the helix $\gamma(t) = (a\cos\theta, a\sin\theta, b\theta)$ with $a > 0$ has curvature $\kappa = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$. Since any two curves in $\mathbb{R}^3$ with the same nonzero curvature function and torsion functions are related by direct isometry, it suffices to show that for any pair of real numbers $\kappa > 0$ and τ, there is a pair $a > 0$ and b such that $\kappa = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$. We see that $\frac{\tau}{\kappa} = \frac{b}{a}$, so we must have $b = \tau c$ and $a = \kappa c$ for some constant c. But then $\kappa = \frac{\kappa c}{\kappa^2 c^2+\tau^2 c^2}$, so $c = \frac{1}{\kappa^2+\tau^2}$. Ergo if $a = \frac{\kappa}{\kappa^2+\tau^2}$ and $b = \frac{\tau}{\tau^2+\kappa^2}$, the helix $\gamma(t) = (a\cos\theta, a\sin\theta, b\theta)$ has constant curvature and torsion κ and τ. We conclude that all curves with constant curvature and torsion are the images of circular helices under direct isometry.

- (2.3.4) We know $\gamma(t)$ is unit-speed and spherical. Without loss of generality, the center of the sphere is $\mathbf{0}$ (because if it isn't we can start by doing a translation). Therefore $\gamma(t)\cdot\gamma(t) = r^2$ where r is the radius of the sphere. Differentiating gives $2\dot{\gamma}\cdot\gamma = 0$. SInce γ is unit-speed, $\dot{\gamma} = \mathbf{t}$, so this says $\mathbf{t}\cdot\gamma = 0$. Differentiating this second relationship again gives

$$\begin{aligned} \mathbf{t}\cdot\mathbf{t} + \kappa\mathbf{n}\cdot\gamma &= 0 \\ 1 + \kappa\mathbf{n}\cdot\gamma &= 0 \\ \mathbf{n}\cdot\gamma &= -\frac{1}{\kappa} \end{aligned}$$

We differentiate the last equality again:

$$\begin{aligned} (-\kappa\mathbf{t} + \tau\mathbf{b})\cdot\gamma + \mathbf{n}\cdot\mathbf{t} &= \frac{\dot{\kappa}}{\kappa^2} \\ \tau\mathbf{b})\cdot\gamma &= \frac{\dot{\kappa}}{\kappa^2} \\ \mathbf{b}\cdot\gamma &= \frac{\dot{\kappa}}{\kappa^2\tau} \end{aligned}$$

Notice this implies that $\gamma(t) = -\frac{1}{\kappa} + \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{n}$ for all t. We differentiate a final time:

$$\begin{aligned} -\tau\mathbf{n}\cdot\gamma + \mathbf{b}\cdot\mathbf{t} &= \frac{d}{dt}\left(\frac{\dot{\kappa}}{\kappa^2\tau}\right) \\ -\tau\left(-\frac{1}{\kappa}\right) &= \frac{d}{dt}\left(\frac{\dot{\kappa}}{\kappa^2\tau}\right) \\ \frac{\tau}{\kappa} &= \frac{d}{dt}\left(\frac{\dot{\kappa}}{\kappa^2\tau}\right) \end{aligned}$$

This completes the forward direction. Conversely, if this equation holds and $\rho = \frac{1}{\kappa}$, $\sigma = \frac{1}{\tau}$, then

$$\begin{aligned}\frac{d}{dt}(\rho^2 + (\dot{\rho}\sigma)^2) &= 2\rho\dot{\rho} + 2\dot{\rho}\sigma\frac{d}{dt}(\dot{\rho}\sigma) \\ &= 2\dot{\rho}\left(\rho + \sigma\frac{d}{dt}(\dot{\rho}\sigma)\right)\end{aligned}$$

But $\rho+\sigma\frac{d}{dt}(\dot{\rho}\sigma) = \frac{1}{\kappa}+\frac{1}{\tau}\frac{d}{dt}\left(\frac{-\dot{\kappa}}{\kappa^2\tau}\right) = 0$ by equation (2.22). So we conclude that $\rho^2+(\dot{\rho}\sigma)^2$ is constant, say r^2. Now we know $\rho^2 + (\dot{\rho}\sigma)^2 = r^2$. So the curve $-\rho\mathbf{n} + (\dot{\rho}\sigma)\mathbf{b}$ lies on the sphere of radius r. Differentiating shows it has the same tangent vector as γ, so up to translation (which just changes the center of the sphere) it is the same curve. Ergo γ is spherical. A computation shows the relationship holds for Viviani's Curve.

- (2.3.5) Let $\gamma(t)$ be our unit-speed curve and $P\mathbf{x} + \mathbf{a}$ be our direct isometry. Then if $\Gamma(t) = P\gamma(t) + \mathbf{a}$, we see that $\Gamma'(t) = P\gamma'(t)$, and since P is orthogonal (and in particular, length-preserving), $||P\gamma'(t)|| = 1$, so $\Gamma(t)$ is unit speed. Since both γ and Γ are unit speed, $\Gamma'(t) = P\gamma'(t)$ is exactly the statement that $\mathbf{T} = P\mathbf{t}$; similarly, $\Gamma''(t) = P\gamma''(t)$, so since $||\gamma''(t)|| = ||P\gamma''(t)||$, we see that $N = \frac{\Gamma''(t)}{||\Gamma''(t)||} = \frac{P\gamma''(t)}{||\gamma''(t)}|| = P\mathbf{n}$. Then $P\mathbf{t} \times P\mathbf{n} = \det(P)P(\mathbf{t} \times \mathbf{n}) = P\mathbf{b}$, so $\mathbf{B} = P\mathbf{b}$.

  (To see the last fact, let P be orthogonal and let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the orthonormal columns of P. Then if P is direct, we have $1 = \det(P) = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$, from the standard algorithm for computing the derivative by going down the third column. So in particular the unit vector $\mathbf{v}_1 \times \mathbf{v}_2$ is $\mathbf{v}_3$, and the columns of P form a right-handed orthonormal system. Then if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$, we can compute the cross-product $P\mathbf{a} \times P\mathbf{c} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \times (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3) = (a_1b_2 - b_1a_2)\mathbf{v}_3 + (a_3b_1 - b_3a_1)\mathbf{v}_2 + (a_2b_3 - b_2a_3)\mathbf{v}_3 = P(\mathbf{a} \times \mathbf{b})$.

- (2.3.13) Recall that a curve is a generalized helix if it makes a fixed angle with some unit vector. Out curve is $\gamma(t) = (e^{\lambda t}\cos t, e^{\lambda t}\sin t, e^{\lambda t})$, with tangent vector $\dot{\gamma}(t) = (\lambda e^{\lambda t}\cos t - e^{\lambda t}\sin t, \lambda e^{\lambda t}\sin t + e^{\lambda t}\cos t, \lambda e^{\lambda t}$. Note that this vector has length $\sqrt{2\lambda^2 + 1}e^{\lambda t}$. Now if $\mathbf{u} = (0, 0, 1)$, then $\dot{\gamma}(t) \cdot \mathbf{u} = \lambda e^{\lambda t}$, so if θ is the angle between $\dot{\gamma}(t)$ and $\mathbf{u}$, then $\cos\theta = \frac{\sqrt{2\lambda^2+1}}{\lambda}$. We conclude θ is constant.

  We also need to check the curvature of γ is nonzero. But note that $\ddot{\gamma}(t) = ((\lambda^2 - 1)e^{\lambda t}\cos t - 2\lambda e^{\lambda t}\sin t, (\lambda^2 - 1)e^{\lambda t}\sin t + 2\lambda e^{\lambda t}\cos t, \lambda^2 e^{\lambda t})$, so the first entry of $\ddot{\gamma} \times \dot{\gamma}$ is $-e^{\lambda t}(\lambda\sin t + 3\lambda^2\cos t)$ and the second is $(\lambda e^{\lambda t}(\lambda\cos t + 3\lambda^2\sin t)$. These two terms cannot be zero simultaneously, so $\ddot{\gamma} \times \dot{\gamma}$ is not the zero vector for any t. Hence γ has nonzero curvature anywhere.